On Cyclical Cauchy Sequences of Cyclically Proximal Sets

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1. Introduction and Preliminaries

In [9] Rafael introduced the notation called proximally complete pair of subsets of a metric space, which weakens the notion of *UC* property and cyclical completeness introduced by Karpagam [5] in the theory of Best proximity points. In [9] the authors also shown that every pair of non-empty closed convex subsets of a uniformly convex banach space (or boundedly compact subsets of a metric space) is proximally complete. In [8] the cyclical proximal property says that if there exists $x_i \in A_i$, for $1 \le i \le p$ such that $x_i = x_{i+p}$ for all $i = 1, 2, \cdots, p$ whenever $\| x_i - x_{i+1} \| = d(A_i, A_{i+1})$.

For a pair of subsets (A_i, A_{i+1}), for $i = 0, 1, \ldots p-1$, where $A_p = A_0$.

$$\text{Let} \quad A_{i+1}^0 = \{ y \in A_{i+1} : d(x, y) = d(A_i, A_{i+1}) \text{ for some } x \in A_i \text{ and } d(y, z) = d(A_{i+1}, A_{i+2}) \text{ for some } z \in A_{i+2} \}$$

Definition 1.1

Let $A_0, A_1, \ldots, A_{p-1}$ be a non-empty subsets of a metric space *X*.

A sequence $\{x_n\}_{n=0}^{\infty}$ in $\bigcup_{i=0}^{p-1} A_i$, with

$x_1 \in A_1, \ldots x_{pn} \in A_p, x_{pn+1} \in A_1 \ldots, x_{p(n+1)-1} \in A_{p-1}$

for $n \ge 0$, is said to be a cyclical Cauchy sequence iff for each pair (A_i, A_{i+1}) and any $\varepsilon > 0$ there exists an $n \ge \mathbb{N}$ such that

$d(x_{pk_1}, x_{pk_2+1}) < d(A_i, A_{i+1}) + \varepsilon$ for $k_1, k_2 \ge N$.

Definition 1.2

The *p*-sets $A_0, A_1, \ldots, A_{p-1}$ of metric space is proximally complete iff for every cyclically Cauchy sequence $\{x_n\}_{n=0}^{\infty} \in \bigcup_{i=0}^{p-1} A_i$, the sequence $\{x_{pn}\}, \{x_{pn+1}\}$, ..., $\{x_{p(n+1)-1}\}$ have convergent subsequences in $A_0, A_1, \ldots, A_{p-1}$ respectively.

Definition 1.3

[6] Let (*X*, *d*) be a metric space and let $A_1, A_2, \ldots, A_p$ be non-empty subsets of *X*. If $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is a *p*-cyclic non-expansive maping, then $d(A_i, A_{i+1}) = d(A_{i+1}, A_{i+2}) = \ldots = d(A_1, A_2)$ for $i = 1, 2, \ldots, p$.

Definition 1.4

[8] The non-empty subsets $A_1, A_2, \ldots, A_p$ of a metric space X said to satisfy cyclical proximal property if there exists $x_1 \in A_i$ for all $1 \le i \le p$ such that $x_i = x_{i+p}$ for all $i = 1, 2, \ldots, p$ whenever $\| x_i - x_{i+1} \| = d(A_i, A_{i+1})$.

Lemma 1.5

[1] *Let A be a non-empty closed and convex subset and B a non-empty and closed subset of a uniformly convex Banach space. Let* $\{x_n\}$ and $\{z_n\}$ be *sequence in A and* $\{y_n\}$ *be a sequence in B satisfying.*

i) $\| z_n - y_n \| \to d(A, B)$;

ii) *For every* $\varepsilon > 0$ *there exists* $N_0 \, \varepsilon \, \mathbb{N}$ such that for all $m > n \ge N_0$, $\| x_m - y_n \| \le d(A, B) + \varepsilon$. *Then for every* $\varepsilon > 0$, *there exists* N_1 *such that* $\| x_m - z_n \| \le \varepsilon$ for all $m > n \ge N_1$.

Definition 1.6

[7] Let A and B be non-empty subsets of a metric space *X*. *(A,B)* is said to satisfy property UC iff whenever $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n\to\infty} d(x_n, y_n) = d(A, B)$ and $\lim_{n\to\infty} d(z_n, y_n) = d(A, B)$, then $\lim_{n\to\infty} d(x_n, z_n) = 0$.

Lemma 1.7

Every cyclical Cauchy sequence is bounded.

Proof

Let $\{x_n\}_{n=0}^{\infty}$ be a cyclical Cauchy sequence in $\bigcup_{i=0}^{p-1} A_i$. Therefore, there exists $N \, \varepsilon \, \mathbb{N}$ such that

$$d(x_{pn}, x_{pN+1}) < d(A_i, A_{i+1}) + 1, \text{ for all } n \ge \mathbb{N}.$$

Therefore, $x_{pn} \in B(x_{pN+1}, r)$ for all $n \ge \mathbb{N}$, where

$$r = \max\{d(x_p, x_{pN+1}), d(x_{2p}, x_{pN+1}), \ldots, d(x_{pN}, x_{pN+1}), d(A_i, A_{i+1}) + 1\}$$

Then $d(x_{pn}, x_{pN+1}) \le r$ for $n \in \mathbb{N}$.

Which implies $\{x_{pn}\}$ is bounded.

Similarly $\{x_{pn+1}\}, \{x_{pn+2}\}, \ldots, \{x_{p(n+1)-1}\}$ are also bounded.

2. Main Results

Theorem 2.1

Let (A_i, A_{i+1}) *be a proximally complete pair in a metric space X. Therefore* A_i^0 *is non-empty iff there exists a cyclical Cauchy sequence in* $\bigcup_{i=0}^{p-1} A_i$.

Proof

Let $\{x_n\}$ be a cyclical cauchy sequence: then there exist $\{x_{pn_k}\}$, $\{x_{pm_k+1}\}$, $\{x_{pi_k+2}\}$, ..., $\{x_{p(jk+1)-1}\}$ convergent subsequences of $\{x_{pn}\}$, $\{x_{pn+1}\}$, $\{x_{pn+2}\}$, ..., $\{x_{p(n+1)-1}\}$ converging to $x_0 \in A_0, x_1 \in A_1, x_2 \in A_2, ...,$ $x_{p-1} \in A_{p-1}$ respectively. Hence

$$d(A_0, A_1) \le d(x_0, x_1) = \lim_{k\to\infty} d(x_{pn_k}, x_{pm_k+1}) = d(A_0, A_1)$$

and

$$d(A_1, A_2) \le d(x_1, x_2) = \lim_{k\to\infty} d(x_{pm_k+1}, x_{pi_k+2}) = d(A_1, A_2)$$

Therefore $x_1 \in A_1^0$.

Similarly $x_i \in A_i^0$ for all i = 0, 1,..., p–1.

Theorem 2.2

Let $A_0, A_1, \ldots, A_{p-1}$ *be subsets of a metric space X. If* (A_i, A_{i+1}) *is proximally complete, then* $A_i^0, i = 0,1,..., p-1$ *are closed subsets of X.*

Proof

Let $x_n^1 \in A_1$ such that $x_n^1 \to x \in X$, $x_n^2 \in A_2, ..., x_n^p \in A_p, x_n^{p-1} \in A_{p-1}$

such that

$$d(x_n^1, x_n^2) = d(A_1, A_2),\ d(x_n^2, x_n^3) = d(A_2, A_3), ..., d(x_n^{p-1}, x_n^p) = d(A_{p-1}, A_0)$$

For $n \in \mathbb{N}$,

$$y_n = \begin{cases} x_m^1, \text{for } n = pm+1 \\ x_m^2, \text{for } n = pm+2 \\ \vdots \\ x_m^p, \text{for } n = pm \end{cases}$$

Then

$$d(y_{pn}, y_{pm+1}) = d(x_n^p, x_m^1)$$
$$\leq d(x_m^1, x) + d(x, x_n^1) + d(x_n^1, x_n^p)$$

which tends to $d(A_0, A_1)$ and

$$d(y_{pn+1}, y_{pm+2}) = d(x_n^1, x_m^2)$$
$$\leq d(x_n^1, x) + d(x, x_m^1) + d(x_m^1, x_m^2)$$

which tends to $d(A_1, A_2)$, as $m, n \to \infty$.

Hence $\{x_n\}$ is a cyclical Cauchy sequence. Since (A_i, A_{i+1}) is proximally complete, $\{x_n^1\}, \{x_n^2\}, \{x_n^3\}, \ldots, \{x_n^{p-1}\}, \{x_n^p\}$ have convergent subsequences which converges to $x_1 \in A_1, \ldots, x_{p-1} \in A_{p-1}, x_p \in A_p$ respectively.

Hence $x = x_1$, so $d(x_0, x) = d(A_0, A_1)$ and $d(x, x_2) = d(A_1, A_2)$ which implies A_1^0 is closed. Similarly A_i^0 for $i = 0, 1, \ldots, p$ are closed.

Theorem 2.3

Any non empty, closed and convex pair (A_i, A_{i+1}) in a uniformly convex Banach space is proximally complete. Furthermore, for any cyclical Cauchy sequence $\{x_n\}$, sequences $\{x_{pn}\}, \{x_{pn+1}\}, \{x_{pn+2}\}, \ldots, \{x_{p(n+1)-1}\}$ converges to $x_0, x_1, x_2, \ldots, x_{p-1}$ respectively, with $d(x_0, x_1) = d(x_1, x_2) = d(x_2, x_3) = \ldots$ $= d(x_{p-1}, x_p) = d(A_i, A_{i+1})$.

Proof

Let $\{x_n\}$ be a cyclical Cauchy sequence in $\bigcup_{i=0}^{p-1} A_i$. Suppose $\{x_{pn}\}$ is not a Cauchy sequence. Therefore, there exists $\varepsilon_0 > 0$ and subsequences $\{x_{pnk}\}$ and $\{x_{pmk}\}$ of $\{x_{pn}\}$ such that $d(x_{pn_k}, x_{pm_k}) \geq \varepsilon_0$ for all $k \in \mathbb{N}$.

One can also observe that

$$d(x_{pn_k}, x_{p_k+1}) \to d(A_0, A_1) \text{ and } d(x_{pm_k}, x_{p_k+1}) \to d(A_0, A_1) \text{ as } k \to \infty.$$

Recalling Lemma 1.5, we reach the contradiction that there exist $N_1 \in \mathbb{N}$ such that $d(x_{pn_k}, x_{pm_k}) < \varepsilon_0$ for all $k \geq \mathbb{N}$.

Hence $\{x_{pn}\}$ converges to a point $x_0 \in A_0$.

Similarly $x_{pn+1} \to x_1 \in A_1, x_{pn+2} \to x_2 \in A_2, \ldots, x_{p(n+1)-1} \to x_{p-1} \in A_{p-1}$

and that

$$d(x_0, x_1) = \lim_{n\to\infty} d(x_{pn}, x_{pn+1}) = d(A_0, A_1);$$

$$d(x_1, x_2) = \lim_{n\to\infty} d(x_{pn+1}, x_{pn+2}) = d(A_1, A_2);$$

$$\vdots$$

$$d(x_p, x_{p-1}) = \lim_{n\to\infty} d(x_{pn}, x_{p(n+1)-1}) = d(A_0, A_{p-1}).$$

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